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Shape functions for concave quadrilaterals

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Abstract

Shape functions of convex n-sided polygons can be constructed efficiently and uniquely using computer algebra. Even this does not suffice for the precise analysis of shape changes in anatomical structures where geometrical concavity is germane to biological function. Shape function generation for the three dimensional concave elements, which does not sacrifice continuity, is currently under research. The associated energy density function can be integrated exactly using the divergence theorem.

Keywords: Concave finite elements; Shape function; Computer algebra

1. Introduction

The anatomical as well as anthropological growth analysis of the maxilo-facial frame in Fig. 1 is primarily the study of changes in its concavity features [4]. The nodal locations can be extracted from radiological images. They serve as universally identifiable reference points. The function of a patient's face and jaw is routinely diagnosed clinically by observing the shapes changes associated with these landmarks [2]. This morphometry is quantified by mechanical strain like tensors defined within the maxillo-facial element.

Conventional finite element discretization introduces artificial landmarks which mar the underlying biology. This is evident from the numerical discontinuity observed in the strain tensor field. The macroelement formulation, based on rational polynomial shape functions, does not introduce those anomalies [5]. Shape functions of any convex n-noded macroelements can be derived consistently from the principles of projective geometry. Taig's quadrilaterals, on the other hand, require a heuristic isoparametric assumption [3]. Unfortunately, Wachspress's macroelements can not account for geometrical concavity.

The concavity restriction is alleviated by analytically enforcing continuity of the displacement, slope and curvature tensor fields within a quadrilateral element. In particular, the shape function associated with the concave node is calculated, then the three remaining shape functions are generated by the three patch test requirements [1].

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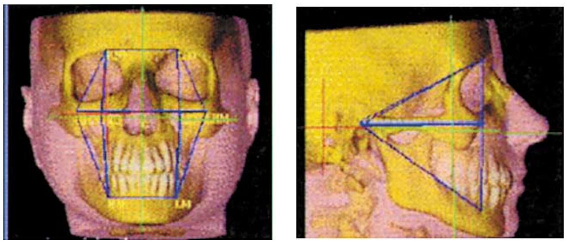

Fig. 1. Maxilo-facial frame.

2. Numerical example

The concave shape calculations are coordinate independent. Thus, for algebraic simplicity the concave node is centered at the origin and the opposite node is placed on the x-axis, see node-1 and node-3 respectively in Fig. 2. Select the following nodal points:

$$\{\{0., 0.\}, \{4., 2.\}, \{-2., 0.\}, \{2., -4.\}\}. \tag{1}$$

The green lines parallel to the ray connecting node-2 and node-3 and the blue lines parallel to the ray connecting node-2 and node-4 define the oblique (η, ξ) coordinate system, see Fig. 2. Two non-dimensional variables τ and ζ are defined to be zero at the concave node, node-1, $\tau = 1$ at the adjacent node, node-3, and $\zeta = 1$ at node-2.

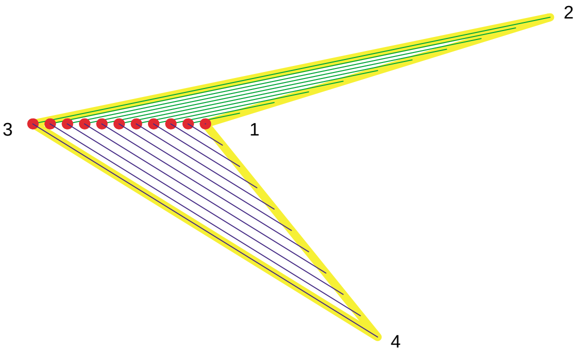

Fig. 2. Rays on a concave element.

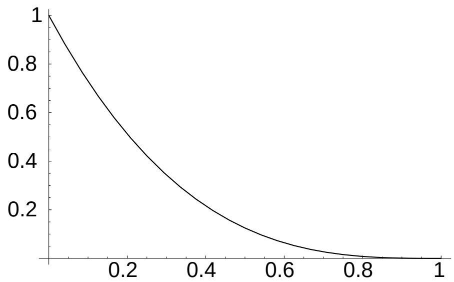

Fig. 3. Assumed profile along the diagonal.

Along the leading diagonal, in this example, a cubic profile is assumed, see Fig. 3. Using the continuity requirements, the rays extending from the axis must have equal displacements, angles and curvatures at the point at which they meet, see the dots in Fig. 2. The perpendicular curvature is assumed to be zero. Thus, the profile on the leading diagonal is given in terms of τ and ζ for every x and y, and along each parallel ray.

For the lower triangle:

$$\{x, y\}_{\text{blue}} = \{-2.\,\tau + 4.\,\xi\,\tau, -4.\,\xi\,\tau\}$$

$$\begin{aligned} u_{\text{blue}} = 1 &- 2.12132\,\xi + 1.83697\,10^{-16}\,\xi^2 + 2.12132\,\xi^3 \\ &- 3\,\tau + 4.24264\,\xi\,\tau - 1.83697\,10^{-16}\,\xi^2\,\tau \\ &- 2.24264\,\xi^3\,\tau + 3\,\tau^2 - 2.12132\,\xi\,\tau^2 \\ &- 0.87868\,\xi^3\,\tau^2 - \tau^3 + 1.\,\xi^3\,\tau^3, \end{aligned} \tag{2}$$

for the upper triangle:

$$\{x, y\}_{\text{green}} = \{-2.\,\tau + 6.\,\xi\,\tau, 2.\,\xi\,\tau\}$$

$$\begin{aligned} u_{\text{green}} = 1 &- 2.84605\,\xi + 2.4\,\xi^2 + 0.44605\,\xi^3 \\ &- 3\,\tau + 5.6921\,\xi\,\tau - 2.4\,\xi^2\,\tau \\ &- 1.2921\,\xi^3\,\tau + 3\,\tau^2 - 2.84605\,\xi\,\tau^2 \\ &- 0.15395\,\xi^3\,\tau^2 - \tau^3 + 1.\,\xi^3\,\tau^3. \end{aligned} \tag{3}$$

By definition τ and ζ are functions in x and y

for the lower triangle:

$$\left\{\xi \to \frac{y}{2.\,x + 2.\,y}, \tau \to -0.5\,x - 0.5\,y\right\}, \tag{4}$$

for the upper triangle:

$$\left\{\xi \to \frac{y}{-1.\,x + 3.\,y}, \tau \to -0.5\,x + 1.5\,y\right\}. \tag{5}$$

Finally the shape functions over the lower and upper portions of the concave domain are:

$$\begin{aligned} \Phi_{\text{blue}} = (&8.\,x^3 + 12.\,x^4 + 6.\,x^5 + 1.\,x^6 + 15.5147\,x^2\,y \\ &+ 39.5147\,x^3\,y + 27.8787\,x^4\,y + 6.\,x^5\,y \\ &+ 7.02944\,x\,y^2 + 46.5442\,x^2\,y^2 + 51.5147\,x^3\,y^2 \\ &+ 15.\,x^4\,y^2 + 1.63604\,y^3 + 23.6655\,x\,y^3 \\ &+ 47.0524\,x^2\,y^3 + 19.875\,x^3\,y^3 + 4.63604\,y^4 \\ &+ 21.0754\,x\,y^4 + 14.625\,x^2\,y^4 + 3.65901\,y^5 \\ &+ 5.625\,x\,y^5 + 0.875\,y^6)/(2.\,x + 2.\,y)^3 \end{aligned}$$

$$\begin{aligned} \Phi_{\text{green}} = (&-1.\,x^3 - 1.5\,x^4 - 0.75\,x^5 - 0.125\,x^6 \\ &+ 6.15395\,x^2\,y + 15.154\,x^3\,y + 10.5385\,x^4\,y \\ &+ 2.25\,x^5\,y - 12.3237\,x\,y^2 - 56.5856\,x^2\,y^2 \\ &- 58.9619\,x^3\,y^2 - 16.875\,x^4\,y^2 + 9.0316\,y^3 \\ &+ 93.0027\,x\,y^3 + 164.04\,x^2\,y^3 + 67.375\,x^3\,y^3 \\ &- 57.3948\,y^4 - 226.676\,x\,y^4 - 150.75\,x^2\,y^4 \\ &+ 124.271\,y^5 + 178.875\,x\,y^5 \\ &- 87.75\,y^6)/(-1.\,x + 3.\,y)^3 \end{aligned}$$

Notice that the result is a rational polynomial in x and y. The computer algebra environment *Mathematica* was employed to carry out the algebraic manipulations.

3. Conclusions

The lateral view of the maxillo-facial frame is a concave polygon as shown in the example. Standard finite element text books such as Zienkiewicz et al. do not cite any reference to concave elements [6]. For this unconventional formulation computer algebra is an indispensable tool. It is anticipated that this first application of the presented finite element formulation will be especially useful to maxillo-facial surgeons.

References

[1] Irons B, Ahmad S. Techniques of Finite Elements. John Wiley and Sons, New York, 1985.
[2] Moss ML, Salentijn L. The primary role of the functional matrix in facial growth. Angle Orthod 1969;55:566–574.
[3] Taig IC. Structural analysis by the matrix displacement method. Engl Electric Aviation 1961;(S017):566–574.

[4] Treil J, Madrid C, Jaeger M, Casteigt J, Borianne Ph. Biometrie tridimensionnelle maxillo-faciale. Biom Hum Anthropol 1997;15(1.2):65–73.
[5] Wachspress EL. A Rational Finite Element Basis. Academic Press, New York, 1975.
[6] Zienkiewicz OC, Taylor RL. The Finite Element Method. McGraw-Hill, New York, 3rd edition, 1977.