Modeling of unbounded media with concave finite elements using the cloning algorithm

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Abstract

Contamination in computed numerical responses is unacceptable for the modeling of unbounded soil. The wave absorbing boundary is necessarily concave since the structure makes an indentation in the semi-infinite domain. Shape functions for concave elements require a higher order formulation than is required for convex shapes. Nevertheless, a smooth, integrable representation can be constructed.

Keywords: Shape function; unbounded-media; soil-structure interaction

1 Introduction

Analysis of wave propagation in unbounded media requires the analysis of an indented semi-infinite domain, figure 1. The boundaries of the finite element mesh must as parallel as possible to the indentation to avoid interfering with the wave propagation. The discontinuity between elements in finite element discretization introduces spurious waves which mar the underlying kinematics.

A formulation applicable to polygons with any number of sides, based on rational polynomial shape functions, does not require tessellation [6]. Unfortunately the rational polynomial formulation is insufficient for domains which exhibit geometrical concavity. The formulation is based on projective geometry and no perspective transformation of any n-dimensional convex shape can result in a 2-D concave shape [3]. Standard finite element text books such as Zienkiewicz et al. do not cite any reference to displacement based concave elements [7]

2 Soil-structure interaction using the cloning algorithm

The cloning algorithm was conceived to generate (static) stiffness, mass and dynamic stiffness matrices for unbounded media [2]. The accuracy of the resulting matrices hinges upon a concavity requirement for the cells. The outgoing wave behavior is overshadowed by spurious waves generated between the artificially constructed convex sub-element boundaries [4]. The proposed concave cells allow for scaling and the resulting non-dimensional frequency should satisfactorily capture the outgoing wave effects.

A general method for finding interpolations within a polygon is considered. Given the shape function requirements of smoothness, boundedness and reproduction of constant and linear fields, a interpolation within a concave element can be constructed.

A shape function must be zero valued at every node and boundary except those adjacent to it. Let $s_{ij}(x, y)$ be a function which is zero along a boundary segment $\bar{ij}$. A function which is zero valued along every boundary except the adjacent boundaries $\bar{kl}$ and $\bar{lm}$ is:

$$r_k(x,y) = \prod_{i \neq k \ \& \ j \neq k} s_{ij}(x,y). \tag{1}$$

A shape function which satisfies the constancy requirement can be constructed simply:

$$N_i(x,y) = \frac{k_i r_i(x,y)}{\sum k_j r_j(x,y)} \quad \text{where} \quad \sum N_i(x,y) = 1. \tag{2}$$

Where k_i are constants. For a rational polynomial formulation, applicable to any convex polygon, the functions $s_{ij}(x,y)$ which are zero valued along a boundary segment are linear.

A series of ellipses can be constructed such that a function is zero between two points. In figure 2, the ellipse which passes through the point $\{-c, r^*\}$ has the following equation in $\{x,y\}$:

$$\frac{x^2}{\frac{1}{4}(r^* + \sqrt{r^{*2} + 4c^2})^2} + \frac{y^2}{\frac{1}{4}(r + \sqrt{r^{*2} + 4c^2})^2 - c^2} = 1 \tag{3}$$

The resulting function for r in terms of x and y is zero valued from point a to b and linear along the lines normal to the x-axis which pass through the endpoints. Solving for $r(x,y)$ and simplifying:

$$r^*(x,y)^2 = \frac{y^2(x^2-c^2)+(x^2-c^2)^2 \pm \sqrt{(y^2(x^2-c^2)+(x^2-c^2)^2)^2-4x^2(y^2)^2c^2}}{2x^2} \tag{4}$$

Choosing the branch which satisfies the zero condition and is positive over the domain and simplifying further results in the following function:

$$r^*(x,y) = \sqrt{\frac{(x^2-c^2)^2+(c^2+x^2)y^2+(x^2-c^2)\sqrt{(x^2+y^2)^2+c^2\,(c^2+2\,(y^2-x^2))}}{2\,x^2}} \tag{5}$$

The function is decidedly nonlinear.

Interpolations constructed on the bases of such elliptical contours can be used to describe concave domains and consequently analyze wave absorbing boundaries, see figure 3. The applicability of approximations to the shape functions for the concave element is tested by comparison to the overall behavior and relative to the boundary element Green's functions [1, 5].

Method	**Weakness**	**Strength**
Triangular or quadrilateral mesh	• Requires mesh • Not smooth	• Polynomial
Boundary element:	• Needs field equation solution • Results along the boundary are approximate	• No mesh
Wachspress element:	• Rational polynomial • Convex shapes only	• No mesh • Smooth
Proposed general element:	• Irrational polynomial	• No mesh • Smooth • Concave & Convex

3 Conclusions

The wave propagation within the wave absorbing boundary is central to the study of soil-structure interaction. For a building in unbounded media, such as the earth, such a boundary is necessarily concave. The concave elements are algebraically more complicated then their convex analog. Consequently, computational tools including computer algebra are indispensable to the creation of a generic formulation which applies to any concave shape.

References

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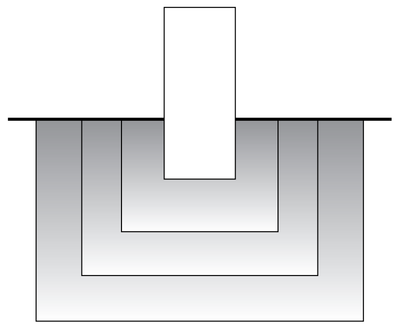

Figure 1: Structure in a semi-infinite soil

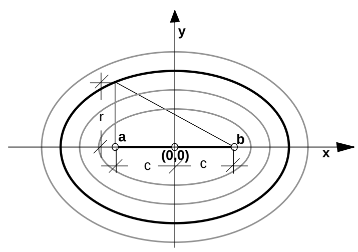

Figure 2: Zero valued function from a to b

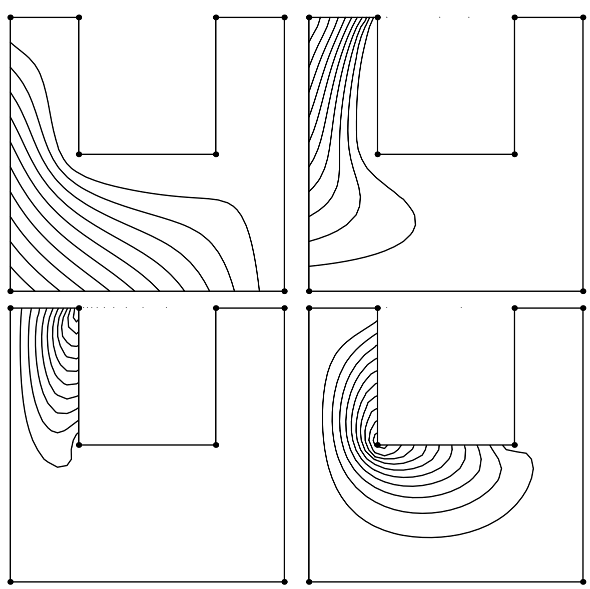

Figure 3: Selected shape function for a concave element